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Decay estimates of functions through singular extensions of vector-valued Laplace transforms [☆]

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ABSTRACT

Let X be a Banach space and let $f \in L^\infty(\mathbb{R}^+; X)$ whose Laplace transform extends analytically to some region containing $i\mathbb{R} \setminus \{0\}$, possibly having a pole at the origin. In this paper, we give estimates of the decay of certain slight suitable modification of f in terms of the growth of its Laplace transform along the imaginary axis. This technique is applied to obtain decay estimates of smooth orbits of bounded C_0 -semigroups whose infinitesimal generators have an arbitrary finite boundary spectrum. These results are close to those given recently by C.J.K. Batty and T. Duyckaerts.

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1. Introduction

Stability theory of C_0 -semigroups of operators on a Banach space has received a lot of attention in recent times. The study of the asymptotic behaviour of these families is important from the point of view of the theory of partial and abstract ordinary differential equations since C_0 -semigroups appear as natural solutions of well-posed abstract Cauchy problems. There is a wide range of results about stability, of a given C_0 -semigroup or some of its orbits, under assumptions of very different nature. Recall that, for a C_0 -semigroup $(T(t))_{t \geq 0}$ on a Banach space X , an orbit $T(\cdot)x$ ($x \in X$) is called stable if $T(t)x \rightarrow 0$ as $t \rightarrow +\infty$. For a fairly complete overview on the techniques used and the results obtained on this topic, see [6,10].

In the last twenty years, many authors have approached the issue of determining the decay rate of stable orbits. Motivated by applications to damped wave equations and many other hyperbolic problems, great progress in the problem of getting decay estimates has been achieved in the case that the infinitesimal generator of the C_0 -semigroup has empty boundary spectrum (see [5,7,9,17,18] and references therein). It turns out from these results that the growth of the resolvent on the imaginary axis determines the rate of decay of smooth orbits of the semigroup. Recently, a unified approach to this statement has been given by Batty and Duyckaerts:

Theorem 1.1. (See [7, Theorem 1.5].) *Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup on a Banach space X . Let A be its generator and assume that $\sigma(A) \cap i\mathbb{R} = \emptyset$. Let $k \in \mathbb{N}$. Then, there are constants $C_k, T_k > 0$ such that*

$$\|T(t)(1 - A)^{-k}\| \leq \frac{C_k}{(M_{\log}^{-1}(t/C_k))^k}, \quad \forall t \geq T_k,$$

where

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$$M(\xi) := \sup_{1 \leq |\tau| \leq \xi} \|(i\tau - A)^{-1}\|, \quad \xi \geq 1, \quad (1.1)$$

$$M_{\log}(\xi) := M(\xi) \log((1 + M(\xi))(1 + \xi)), \quad \xi \geq 1. \quad (1.2)$$

In the same article [7], it is conjectured that the logarithmic correction considered in (1.2) is necessary when X is a general Banach space but it can be dropped if X is a Hilbert space. In the recent paper [8], A. Borichev and Y. Tomilov confirm this conjecture for polynomially growing M . In addition, they show that Theorem 1.1 is sharp.

The proof of Theorem 1.1 is based on a classical contour integral method initiated by Newman and Korevaar (see [20,16]). Using this technique, C.J.K. Batty and T. Duyckaerts also estimate the decay of some Cesaro means of a bounded vector-valued function whose Laplace transform extends to a suitable region containing the imaginary axis [7, Theorem 4.1]. They conclude the work by considering the case of semigroups whose associated boundary spectrum is at most finite [7, Proposition 4.3].

The fruitful connection between vector-valued Laplace transforms, semigroup theory and evolution equations has been developed by many authors. For a function $f \in L^\infty(\mathbb{R}^+; X)$, we denote by \hat{f} the Laplace transform of f which is given by

$$\hat{f}(z) = \int_0^\infty e^{-zt} f(t) dt, \quad \Re z > 0.$$

We recommend the monograph [4] and references therein for the state of the art.

In the main result of the present paper (Theorem 2.3), we obtain estimates for the rate of decay of certain slight modifications of a given function $f \in L^\infty(\mathbb{R}^+; X)$ in terms of the growth of \hat{f} on the imaginary axis. In more detail we shall assume that \hat{f} has an analytic extension to some region containing $i\mathbb{R} \setminus \{0\}$, where it satisfies *suitable* bounds. Observe that \hat{f} might have a pole at the origin. Then, we estimate the decay of $(e_1 * e_1 - e_1) \circ f$ where $e_1(t) := e^{-t}$ for $t \in \mathbb{R}^+ = (0, +\infty)$, $*$ is the usual convolution in $L^1(\mathbb{R}^+)$ given by

$$g * h(t) := \int_0^t g(t-s)h(s) ds, \quad t > 0, \quad g, h \in L^1(\mathbb{R}^+),$$

and \circ denotes the convolution product defined as

$$g \circ f(t) := \int_t^\infty g(s-t)f(s) ds, \quad t > 0, \quad g \in L^1(\mathbb{R}^+), \quad f \in L^\infty(\mathbb{R}^+; X).$$

Note that this result is in the spirit of [7, Theorem 4.1] and [2, Proposition 1.1].

In the third section, Theorem 2.3 is applied to C_0 -semigroups under the assumption that the boundary spectrum of their infinitesimal generators is empty or consists of the origin. Such semigroups appear in applications to wave equations (see [9,12,13] and references therein). We estimate the decay of certain orbits in terms of the norm of the resolvent operator along the imaginary axis (Proposition 3.1). This result is in the spirit of the above-mentioned Theorem 1.1.

Finally, in Theorem 3.4 we show that similar estimates can be given for the rate of decay of certain orbits of bounded semigroups having arbitrary finite boundary spectrum. This result extends Proposition 3.1 and completes [7, Proposition 4.3]. The added interest of this result is that such estimates are given in an explicit way in terms of the resolvent operator along the imaginary axis.

2. The decay rate of functions

This section is devoted to state and prove Theorem 2.3. To simplify the understanding, we include first some technical lemmata.

Throughout the following, we denote $e_z(t) := e^{-zt}$ for every $z \in \mathbb{C}$ and $t \in \mathbb{R}^+$. Notice that $e_z \in L^1(\mathbb{R}^+)$ whenever z belongs to the open right-half plane \mathbb{C}^+ and $\|e_z\|_1 = 1/\Re z$. It is also well known that $(e_z)_{z \in \mathbb{C}}$ verifies the resolvent identity

$$(z - \omega)(e_z * e_\omega) = e_\omega - e_z, \quad z, \omega \in \mathbb{C}.$$

On the other hand, given some $f \in L^\infty(\mathbb{R}^+; X)$ and $g \in L^1(\mathbb{R}^+)$, let $g \circ f \in L^\infty(\mathbb{R}^+; X)$ denote the convolution product given by

$$g \circ f(t) := \int_t^\infty g(s-t)f(s) ds, \quad t > 0,$$

where the integral is understood in the sense of Bochner. This product is the *adjoint convolution* to the usual one in $L^1(\mathbb{R}^+)$ in the sense that for any $g, h \in L^1(\mathbb{R}^+)$ we have

$$\int_0^\infty (g * h)(t) f(t) dt = \int_0^\infty h(t) (g \circ f)(t) dt, \quad f \in L^\infty(\mathbb{R}^+; X).$$

However, this product \circ is not commutative nor associative. Some of the properties of this convolution that we will use in the sequel are the following (see for instance [19,11]):

- (i) $g \circ (h \circ f) = (g * h) \circ f = h \circ (g \circ f)$ for f, g, h as above.
- (ii) $e_z \circ e_\omega = \frac{1}{z+\omega} e_\omega$ for every $z, \omega \in \mathbb{C}^+$.

Lemma 2.1. Let $f \in L^\infty(\mathbb{R}^+; X)$. Then,

$$(e_1 - ze_z * e_1) \circ f = e_z \circ (f - e_1 \circ f), \quad z \in \mathbb{C}^+. \quad (2.1)$$

Proof. Taking into account that $(z-1)(e_z * e_1) = e_1 - e_z$ for all $z \in \mathbb{C}^+$, it is readily seen that

$$(e_1 - ze_z * e_1) \circ f = e_1 \circ f - z(e_z * e_1) \circ f = e_z \circ f - (e_z * e_1) \circ f.$$

Now, the claim follows trivially from the basic properties of \circ mentioned above. \square

Lemma 2.2. Let $f \in L^\infty(\mathbb{R}^+; X)$ be such that its Laplace transform \hat{f} extends to an analytic function in some region Ω containing \mathbb{C}^+ . Then, the function

$$z \mapsto (e_1 - ze_z * e_1) \circ f(t), \quad \mathbb{C}^+ \rightarrow X$$

also extends analytically to Ω for every $t > 0$. Moreover, if $z \in \Omega \setminus \{1\}$ and $t > 0$ then

$$(e_1 - ze_z * e_1) \circ f(t) = \frac{1}{1-z} (zg_z(t) + e_1 \circ f(t)) + \frac{z}{1-z} e^{zt} \hat{f}(z) \quad (2.2)$$

where $g_z(t)$ denotes the entire function given by

$$g_z(t) := e^{tz} \int_0^t e^{-sz} f(s) ds, \quad z \in \mathbb{C}.$$

Proof. Let $t > 0$ be fixed and $z \in \mathbb{C}^+$. Notice that

$$e_z \circ f(t) = \int_t^\infty e^{-z(s-t)} f(s) ds = e^{zt} \hat{f}(z) - g_z(t).$$

By (2.1) and the expression of $e_z \circ f$ above, we get that

$$(e_1 - ze_z * e_1) \circ f(t) = e^{zt} \hat{f}(z) - g_z(t) - e_z \circ (e_1 \circ f)(t).$$

It is also straightforward from the definition of \circ that

$$e_z \circ (e_1 \circ f)(t) = e^{zt} \int_0^\infty e^{-sz} (e_1 \circ f)(s) ds - e^{zt} \int_0^t e^{-sz} (e_1 \circ f)(s) ds.$$

Considering the integral expression of $e_1 \circ f$ and applying Fubini Theorem in both integrals, we obtain that if $z \neq 1$ then

$$e_z \circ (e_1 \circ f)(t) = \frac{e^{zt}}{1-z} \hat{f}(z) - \frac{1}{1-z} g_z(t) - \frac{1}{1-z} (e_1 \circ f)(t).$$

From this, we get that for any $z \in \mathbb{C}^+ \setminus \{1\}$

$$(e_1 - ze_z * e_1) \circ f(t) = \frac{1}{1-z} (zg_z(t) + e_1 \circ f(t)) + \frac{z}{1-z} e^{zt} \hat{f}(z).$$

Now, observe that the right-hand side of this equality has an analytic extension to the region $\Omega \setminus \{1\}$. Thus, by the identity theorem for analytic functions, we obtain that the function

$$z \mapsto (e_1 - ze_z * e_1) \circ f(t), \quad \mathbb{C}^+ \rightarrow X$$

extends analytically to the region Ω and, also, the last equality holds in $\Omega \setminus \{1\}$, which proves (2.2) and concludes the proof. \square

Given a continuous function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we will denote

$$\Sigma_\mu := \left\{ z \in \mathbb{C} : \Re z > -\frac{1}{\mu(|\Im z|)} \right\}.$$

In Theorem 2.3, we assume that the Laplace transform of a given function admits an analytic extension to a region Σ_μ . This assumption on the Laplace transform is certainly natural and it has been considered in other settings, see for example [11, Section 4] and references therein. The outline of the proof of Theorem 2.3 is inspired by that of Theorem 1.1 above, which is based on the contour integral method introduced by Newman and Korevaar [20,16]. We shall apply a suitable adaptation of this technique, similarly to [1,3] where some singularities are considered.

Theorem 2.3. *Let X be a Banach space and let $f \in L^\infty(\mathbb{R}^+; X)$. Assume that there exists a continuous function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ verifying the following conditions:*

- (i) *The Laplace transform \hat{f} has a holomorphic extension to the region Σ_μ and $\|\hat{f}(z)\| \leq \mu(|\Im z|)$ throughout $\Sigma_\mu \cap \mathbb{C}^-$.*
- (ii) *μ is decreasing on $(0, 1]$ and increasing on $[1, +\infty)$.*

Then, there exist positive constants C and T such that

$$\|(e_1 - e_1 * e_1) \circ f(t)\| \leq C \left(m_{\log}^{-1}(t/4) + \frac{1}{M_{\log}^{-1}(t/4)} + \frac{1}{t} \right), \quad t > T,$$

where m_{\log}^{-1} and M_{\log}^{-1} denote the inverse functions of m_{\log} and M_{\log} , respectively, defined by

$$M_{\log}(\xi) := \mu(\xi) \log((1 + \mu(\xi))(1 + \xi)), \quad \xi \geq 1, \quad (2.3)$$

$$m_{\log}(\xi) := \mu(\xi) \log\left(\frac{1 + \mu(\xi)}{\xi}\right), \quad 0 < \xi \leq 1. \quad (2.4)$$

Proof. Under these assumptions, notice that the functions M_{\log} and m_{\log} are strictly increasing and decreasing, respectively. We are then allowed to consider the inverse functions $M_{\log}^{-1} : [M_{\log}(1), \infty) \rightarrow [1, \infty)$ and $m_{\log}^{-1} : [m_{\log}(1), \infty) \rightarrow (0, 1]$, which satisfy

$$\lim_{t \rightarrow \infty} m_{\log}^{-1}(t) = 0 = \lim_{t \rightarrow \infty} \frac{1}{M_{\log}^{-1}(t)}.$$

For $d > 0$, let γ_d^+ and γ_d^- denote the right- and left-hand half of the circle $|z| = d$, respectively. Let $t > 0$ and let any $R > 1$ and $0 < r < \frac{1}{2}$. Set $\gamma := \gamma_R^+ \cup \gamma_r^+ \cup \gamma'$ where γ' is a path in $\Sigma_\mu \cap \mathbb{C}^-$, which is to be chosen later, so that γ is closed, rectifiable and homotopic to zero. Hence, Cauchy's Theorem yields

$$(e_1 - e_1 * e_1) \circ f(t) = \frac{N_{R,r}}{2\pi i} \int_{\gamma} \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) (e_1 - ze_z * e_1) \circ f(t) \frac{dz}{z-1}$$

where $N_{R,r} := (1 + 1/R^2)^{-1} (1 + r^2)^{-1} \leq 1$. The factors $1 + z^2/R^2$ and $1 + r^2/z^2$ play an important role in order to get suitable upper estimates of the integral above. We will write C to denote any positive constant, which may change from line to line, depending only on $\|f\|_\infty$ and the function μ .

First, we prove that the norm of $(e_1 - e_1 * e_1) \circ f(t)$ is bounded by

$$r + \frac{1}{R-1} + \left\| \int_{\gamma'} \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) \frac{z}{1-z} e^{zt} \hat{f}(z) \frac{dz}{z-1} \right\| \quad (2.5)$$

up to some constant $C > 0$.

Note that if $|z| = R$, $|1 + z^2/R^2| = 2|\Re z|/R$ and $|1 + r^2/z^2| \leq 2$. Taking into account the equality in (2.1), we observe that

$$\|(e_1 - ze_z * e_1) \circ f(t)\| \leq 2\|f\|_\infty \|e_z\|_1 \leq C/\Re z$$

whenever $z \in \mathbb{C}^+$. Therefore,

$$\left\| \int_{\gamma_R^+} \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) (e_1 - ze_z * e_1) \circ f(t) \frac{dz}{z-1} \right\| \leq \frac{C}{R-1}. \quad (2.6)$$

On the other hand, if $|z| = r$ we have $|1 + z^2/R^2| \leq 2$ and $|1 + r^2/z^2| = |1 + z^2/r^2| = 2|\Re z|/r$. In addition, if $z \in \gamma_r^+$,

$$\|(e_1 - ze_z * e_1) \circ f(t)\| \leq \|f\|_\infty \|e_1 - ze_z * e_1\|_1 \leq C \frac{r}{\Re z}$$

and, as a consequence,

$$\left\| \int_{\gamma_r^+} \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) (e_1 - ze_z * e_1) \circ f(t) \frac{dz}{z-1} \right\| \leq Cr. \quad (2.7)$$

Up to now, we then have that the norm of $(e_1 - e_1 * e_1) \circ f(t)$ is bounded by

$$r + \frac{1}{R-1} + \left\| \int_{\gamma'} \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) (e_1 - ze_z * e_1) \circ f(t) \frac{dz}{z-1} \right\|,$$

up to some constant $C > 0$.

Now, we consider the expression of $(e_1 - ze_z * e_1) \circ f(t)$ given by (2.2). Thus, the norm of the last integral is bounded by

$$\left\| \int_{\gamma'} \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) \frac{1}{1-z} (zg_z(t) + e_1 \circ f(t)) \frac{dz}{z-1} \right\| + \left\| \int_{\gamma'} \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) \frac{z}{1-z} e^{zt} \hat{f}(z) \frac{dz}{z-1} \right\|.$$

Denote for simplicity

$$G_t(z) := \left(1 + \frac{z^2}{R^2}\right) \left(1 + \frac{r^2}{z^2}\right) \frac{1}{1-z} (zg_z(t) + e_1 \circ f(t)) \frac{1}{z-1}, \quad z \in \mathbb{C}^-.$$

By Cauchy's Theorem,

$$\int_{\gamma' \cup \gamma_r^-} G_t(z) dz = \int_{\gamma_R^-} G_t(z) dz$$

since both paths $\gamma' \cup \gamma_r^-$ and γ_R^- go from iR to $-iR$. Next, we see that the integrals of G_t along γ_R^- and γ_r^- are bounded similarly to (2.6) and (2.7), respectively. First, it is easy to check that

$$\frac{|\Re z|}{|z|} \|zg_z(t) + e_1 \circ f(t)\| \leq C, \quad z \in \mathbb{C}^-.$$

Thus, acting as in (2.6), we get that

$$\left\| \int_{\gamma_R^-} G_t(z) dz \right\| \leq \frac{C}{R-1}$$

and, analogously to (2.7),

$$\left\| \int_{\gamma_r^-} G_t(z) dz \right\| \leq Cr.$$

Therefore,

$$\left\| \int_{\gamma'} G_t(z) dz \right\| \leq C \left(r + \frac{1}{R-1} \right).$$

Finally, putting together all the estimates above, we get the estimate in (2.5).

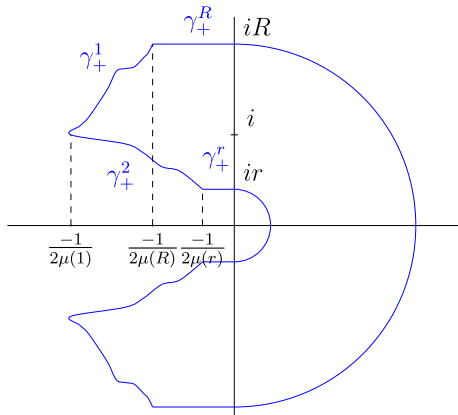
Next, we consider a suitable choice of γ' . In particular, let γ_+ and γ_- to be the two paths in $\Sigma_\mu \cap \mathbb{C}^-$ such that $\gamma' = \gamma_+ \cup \gamma_-$. We take $\gamma_\pm := \gamma_\pm^R \cup \gamma_\pm^1 \cup \gamma_\pm^2 \cup \gamma_\pm^r$ where

$$\begin{aligned} \gamma_\pm^R(s) &= s \pm iR, \quad \frac{-1}{2\mu(R)} \leq s \leq 0, \\ \gamma_\pm^1(\tau) &= \frac{-1}{2\mu(\tau)} \pm i\tau, \quad 1 \leq \tau \leq R, \\ \gamma_\pm^2(\tau) &= \frac{-1}{2\mu(\tau)} \pm i\tau, \quad r \leq \tau \leq 1, \\ \gamma_\pm^r(s) &= s \pm ir, \quad \frac{-1}{2\mu(r)} \leq s \leq 0. \end{aligned}$$

Moreover, we know that

$$\begin{aligned} \|\hat{f}(\gamma_\pm^R(s))\| &\leq \mu(R), \quad \frac{-1}{2\mu(R)} \leq s < 0, \\ \|\hat{f}(\gamma_\pm^1(\tau))\| &\leq \mu(\tau), \quad 1 \leq \tau \leq R, \\ \|\hat{f}(\gamma_\pm^2(\tau))\| &\leq \mu(\tau), \quad r \leq \tau \leq 1, \\ \|\hat{f}(\gamma_\pm^r(s))\| &\leq \mu(r), \quad \frac{-1}{2\mu(r)} \leq s < 0. \end{aligned}$$

We can consider γ' to be piecewise smooth (if it is not, it can be approximated by smooth paths). In particular, γ' remains rectifiable.



We shall now estimate the norm of the integral in (2.5). The factor $z/(z-1)$ is uniformly bounded along γ' . Straightforward calculations show that both factors $1+r^2/z^2$ and $1+z^2/R^2$ are also bounded on γ' by some constant depending only on the value $\mu(1)$. We do not need sharper bounds on these factors in order to get suitable estimates, except for γ_\pm^r .

First, the integral in (2.5) over γ_\pm^R is bounded by

$$C \int_{-(2\mu(R))^{-1}}^0 \mu(R) e^{st} \frac{ds}{|\gamma_\pm^R(s) - 1|} \leq \frac{C}{2R}.$$

On γ_\pm^r , we have $|1+r^2/z^2| \leq 2|\Re z|/|z|$ and $|z-1| \geq 1$, so that

$$\left\| \left(1 + \frac{r^2}{z^2} \right) \frac{z}{z-1} \hat{f}(z) \right\| \leq 2 \frac{|\Re z|}{|z-1|} \|\hat{f}(z)\| \leq 2 \frac{1}{2\mu(r)} \mu(r) = 1.$$

Hence, the integral in (2.5) over γ_{\pm}^r is bounded by

$$C \int_{-(2\mu(r))^{-1}}^0 e^{st} ds \leq \frac{C}{t} (1 - e^{-t/2\mu(r)}) \leq \frac{C}{t}.$$

On the other hand, the integral over γ_{\pm}^1 can be bounded by

$$C \int_1^R \mu(\tau) e^{-t/2\mu(\tau)} d\tau \leq C\mu(R)(R-1)e^{-t/2\mu(R)} \leq \frac{C}{R-1} ((1+\mu(R))^2(1+R)^2 e^{-t/2\mu(R)}).$$

Now, we see that if we consider $t > 4M_{\log}(1)$ and we set $R = M_{\log}^{-1}(t/4)$, exactly as in [7], then $R > 1$ and, moreover,

$$(1 + \mu(R))^2 (1 + R)^2 = e^{2M_{\log}(R)/\mu(R)} = e^{t/2\mu(R)}.$$

Thus, the norm of the integral over γ_{\pm}^1 is bounded by $C/(R-1)$. Finally, we may bound the integral over γ_{\pm}^2 by

$$\begin{aligned} C \int_r^1 \mu(\tau) e^{-t/2\mu(\tau)} \frac{d\tau}{|\gamma_{\pm}^2(\tau) - 1|} &\leq C\mu(r) e^{-t/2\mu(r)} \\ &\leq Cr \left(\frac{(1 + \mu(r))^2}{r^2} e^{-t/2\mu(r)} \right). \end{aligned}$$

Analogously to the preceding case, we notice that if we set $r = m_{\log}^{-1}(t/4)$ then $r < 1/2$ whenever $t > 4m_{\log}(1/2)$. Furthermore,

$$\frac{(1 + \mu(r))^2}{r^2} = e^{2m_{\log}(r)/\mu(r)} = e^{t/2\mu(r)},$$

so that the integral over γ_{\pm}^2 is bounded by Cr .

In conclusion, choosing $R = M_{\log}^{-1}(t/4)$ and $r = m_{\log}^{-1}(t/4)$, the claim of Theorem 2.3 holds for some $T \geq 4 \max\{M_{\log}(1), m_{\log}(1/2)\}$. \square

Remark 2.4. By looking at the estimates in detail, one realizes that the final constant $C > 0$ appearing in Theorem 2.3 is of the form $C = C_{\mu} \|f\|_{\infty}$ where $C_{\mu} > 0$ is some constant depending only on μ .

Remark 2.5. Assumption (ii) in Theorem 2.3 is not strictly necessary in order to get similar decay estimates. In particular, given $f \in L^{\infty}(\mathbb{R}^+; X)$ and a continuous function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that (i) is satisfied, then the function $\nu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined as

$$\nu(t) := \begin{cases} \sup_{t \leq s \leq 1} \mu(s) & \text{if } 0 \leq t \leq 1, \\ \sup_{1 \leq s \leq t} \mu(s) & \text{if } t \geq 1 \end{cases}$$

is continuous and verifies both conditions (i) and (ii). The claim of Theorem 2.3 then holds for M_{\log} and m_{\log} defined in terms of ν .

To illustrate Theorem 2.3, we next consider some particular cases of growth of the Laplace transform near its singularities on the imaginary axis. Some examples for the growth at infinity are considered in [7] and remain valid in this setting. Indeed, set $M \equiv \mu|_{[1, \infty)}$ and let $\alpha, \beta > 0$:

- If $M(\xi) = \beta e^{\alpha\xi}$ then $M_{\log}^{-1}(t) \sim \frac{1}{\alpha} \log(t)$, $t \rightarrow +\infty$.
- If $M(\xi) = \beta(1 + \xi)^{\alpha}$ then $M_{\log}^{-1}(t) \sim C_{\alpha, \beta} \left(\frac{t}{\log(t)}\right)^{\frac{1}{\alpha}}$, $t \rightarrow +\infty$.
- If M is bounded then $M_{\log}^{-1}(t) \sim e^{Ct}$ ($C > 0$), $t \rightarrow +\infty$.

As examples of growth for $m \equiv \mu|_{(0, 1]}$, we can consider the following:

- If $m(\xi) = \beta e^{\alpha\xi^{-1}}$ then $m_{\log}^{-1}(t) \sim \frac{\alpha}{\log(t)}$, $t \rightarrow +\infty$.
- If $m(\xi) = \beta\xi^{-\alpha}$ then $m_{\log}^{-1}(t) \sim C_{\alpha, \beta} \left(\frac{\log(t)}{t}\right)^{\frac{1}{\alpha}}$, $t \rightarrow +\infty$.
- If m is bounded then $m_{\log}^{-1}(t) \sim e^{-Ct}$ ($C > 0$), $t \rightarrow +\infty$.

3. Applications to semigroup theory

From now on, given a Banach space X and a closed linear operator $A : D(A) \subseteq X \rightarrow X$, we denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set of A , respectively. By $\|\cdot\|$, we denote both the norm of X and the operator norm of linear and continuous operators from X to X .

3.1. One point in the boundary spectrum

Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup on X and let A denote its infinitesimal generator. See definitions and properties in [4]. Assume that $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$. Then, the function $\tau \mapsto \|(i\tau - A)^{-1}\|$ is continuous on $\mathbb{R} \setminus \{0\}$.

Let $M : [1, \infty) \rightarrow \mathbb{R}^+$ be the continuous function defined as in (1.1). Consider also the continuous decreasing function given by

$$m(\xi) := \sup_{\xi \leq |\tau| \leq 1} \|(i\tau - A)^{-1}\|, \quad 0 < \xi \leq 1.$$

Now, we define $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as

$$\mu(\xi) := \begin{cases} 2m(\xi) & \text{if } 0 < \xi \leq 1, \\ 2M(\xi) & \text{if } \xi \geq 1. \end{cases} \quad (3.1)$$

Proposition 3.1. Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup on a Banach space X and let A be its generator. Assume that $\sigma(A) \cap i\mathbb{R} \subseteq \{0\}$. Let $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the continuous function defined as in (3.1) above and let M_{\log} and m_{\log} be defined in terms of μ as in (2.3) and (2.4). Then, for any $k \in \mathbb{N}$ there exist positive constants C_k and T_k such that for all $t > T_k$,

$$\|T(t)A^k(1-A)^{-2k}\| \leq C_k \left(m_{\log}^{-1}(t/4k) + \frac{1}{M_{\log}^{-1}(t/4k)} + \frac{k}{t} \right)^k \quad (3.2)$$

where M_{\log}^{-1} and m_{\log}^{-1} denote the inverse functions of M_{\log} and m_{\log} , respectively.

Proof. Since $\|T(t)A^k(1-A)^{-2k}\| \leq \|T(t/k)A(1-A)^{-2}\|^k$, $t \geq 0$, it suffices to prove (3.2) for $k = 1$. Let $x \in X$ such that $\|x\| = 1$ and set $f(t) := T(t)x$ for $t \geq 0$. Since $(1-A)^{-1} - I = A(1-A)^{-1}$, we have that

$$\begin{aligned} T(t)A(1-A)^{-2}x &= T(t)((1-A)^{-2}x - (1-A)^{-1}x) = T(t) \int_0^\infty (e_1 * e_1 - e_1)(s)T(s)x ds \\ &= \int_0^\infty (e_1 * e_1 - e_1)(s)T(t+s)x ds = \int_t^\infty (e_1 * e_1 - e_1)(s-t)f(s) ds \\ &= -(e_1 - e_1 * e_1) \circ f(t). \end{aligned}$$

Now, the claim follows from Theorem 2.3 and Remark 2.4 just by checking that μ satisfies the assumptions in this theorem. It is clear from the definition that μ is decreasing on $(0, 1]$ and increasing on $[1, +\infty)$, as well as that μ is continuous. On the other hand, we know that $i\mathbb{R} \setminus \{0\} \subseteq \rho(A)$ so that we may extend the resolvent operator into the left-half plane by means of standard Neumann series. In particular, if $z \in \Sigma_\mu \cap \mathbb{C}^-$ then $i\Im z \in \rho(A)$ and

$$|i\Im z - z| = |\Re z| \leq \frac{1}{\mu(|\Im z|)} \leq \frac{1}{2\|(i\Im z - A)^{-1}\|}$$

so that $z \in \rho(A)$ and

$$(z - A)^{-1} = \sum_{n=0}^{\infty} (i\Im z - A)^{-(n+1)} (i\Im z - z)^n.$$

Now, it follows easily that $\|(z - A)^{-1}x\| \leq 2\|(i\Im z - A)^{-1}x\| \leq \mu(|\Im z|)$. \square

Remark 3.2. Under the assumptions of Proposition 3.1, the operator $A(1-A)^{-2}$ is a particular case of the functional calculus given by

$$\pi(h)x := \int_0^\infty h(t)T(t)x dt, \quad x \in X, h \in L^1(\mathbb{R}^+). \quad (3.3)$$

Indeed, $A(1 - A)^{-2} = \pi(e_1 * e_1 - e_1)$. Moreover, observe that the function $e_1 * e_1 - e_1 \in L^1(\mathbb{R}^+)$ is of spectral synthesis with respect to $(i\sigma(A)) \cap \mathbb{R}$ since its Fourier transform vanishes at $\{0\}$. Therefore, the Katznelson–Tzafriri type theorem for C_0 -semigroups (see [14,21]) yields

$$\lim_{t \rightarrow \infty} \|T(t)A^k(1 - A)^{-2k}\| = 0 \quad \text{for all } k \in \mathbb{N}.$$

Definitions and details about sets and functions of spectral synthesis can be seen in [14,15,21].

Corollary 3.3. *Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup on a Banach space X generated by an operator B such that $\sigma(B) \cap i\mathbb{R} \subseteq \{ip\}$ for some $p \in \mathbb{R}$. Let $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be defined in terms of $A := B - ip$ as in (3.1). Then, there are constants $C_k, T_k > 0$ such that for all $t \geq T_k$,*

$$\|T(t)(ip - B)^k(1 + ip - B)^{-2k}\| \leq C_k \left(m_{\log}^{-1}(t/4k) + \frac{1}{M_{\log}^{-1}(t/4k)} + \frac{k}{t} \right)^k$$

where M_{\log} and m_{\log} are given by (2.3) and (2.4), respectively.

Proof. The claim follows immediately from Theorem 3.1 by noticing that $(e^{-ipt}T(t))_{t \geq 0}$ is a bounded C_0 -semigroup generated by $B - ip$ such that $\sigma(B - ip) \cap i\mathbb{R} \subseteq \{0\}$. \square

3.2. Arbitrary finite boundary spectrum

Let $(T(t))_{t \geq 0}$ be a bounded C_0 -semigroup on a Banach space X . Assume that the boundary spectrum of its generator A is at most finite, that is, $\sigma(A) \cap i\mathbb{R} \subseteq \bigcup_{j=1}^n \{ip_j\}$ for some $n \in \mathbb{N}$ and some $\bigcup_{j=1}^n \{p_j\} \subseteq \mathbb{R}$.

Under these hypothesis, the bounded operator

$$(1 - A)^{-1} \prod_{j=1}^n ((ip_j - A)(1 - A)^{-1}) \tag{3.4}$$

is a particular case of the functional calculus given by (3.3). Indeed, let

$$g(x) := \frac{1}{1 + ix} \prod_{j=1}^n \frac{ip_j + ix}{1 + ix}, \quad x \in \mathbb{R}.$$

Observe that $g \in C_0(\mathbb{R})$ and vanishes on $i\sigma(A) \cap \mathbb{R}$. Note also that

$$\frac{ip_j + ix}{1 + ix} = 1 - \frac{1}{1 + ix} + \frac{ip_j}{1 + ix} = (\delta - (1 - ip_j)u)^\wedge(x), \quad \forall x \in \mathbb{R},$$

where δ denotes the Dirac measure at the origin and $^\wedge$ denotes now the Fourier transform. Then, the function

$$f := u * \prod_{j=1}^n (\delta - (1 - ip_j)u) \in L^1(\mathbb{R}^+)$$

is such that $\hat{f} = g$. From this and the fact that finite sets of \mathbb{R} are of spectral synthesis, it follows that f is of spectral synthesis with respect to $i\sigma(A) \cap \mathbb{R}$. Furthermore, it is easy to check that the operator in (3.4) is equal to $\pi(f)$. Hence, it follows from the continuous version of the Katznelson–Tzafriri Theorem above mentioned that

$$\lim_{t \rightarrow +\infty} \left\| T(t)(1 - A)^{-1} \prod_{j=1}^n ((ip_j - A)(1 - A)^{-1}) \right\| = 0.$$

In the following Theorem 3.4, we estimate this decay in terms of certain functions M and m , similar to those considered in the last case, defined by means of the resolvent operator along the imaginary axis when avoiding the possible singularities ip_j .

In order to define these functions, assume without loss of generality that $n \geq 2$ and $p_l < p_j$ whenever $1 \leq l < j \leq n$. Let

$$d := \min_{1 \leq j < n} \left\{ \frac{p_{j+1} - p_j}{2} \right\} \quad \text{and} \quad D \geq \max\{|p_n + d|, |p_1 - d|\}.$$

Let $K := [-D, D] \setminus \bigcup_{j=1}^n (p_j - d, p_j + d)$ and denote

$$m_K := \sup_{\tau \in K} \|(i\tau - A)^{-1}\| < \infty.$$

Now, let M be the continuous positive increasing function given by

$$M(\xi) := \sup_{D \leq |\tau| \leq \xi} \{ \|(i\tau - A)^{-1}\|, m_K \}, \quad \xi \geq D.$$

Also, for each $j = 1, \dots, n$, we define

$$m_j(\xi) := \sup_{\xi \leq |\tau - p_j| \leq d} \{ \|(i\tau - A)^{-1}\|, m_K \}, \quad 0 < \xi \leq d,$$

and set

$$m(\xi) := \sup_{1 \leq j \leq n} m_j(\xi), \quad 0 < \xi \leq d,$$

which is continuous, positive and decreasing. Moreover, we define

$$M_{\log}(\xi) := M(\xi) \log((1 + M(\xi))(1 + \xi)), \quad \xi \geq D, \quad (3.5)$$

and

$$m_{\log}(\xi) := m(\xi) \log\left(\frac{1 + m(\xi)}{\xi}\right), \quad 0 < \xi \leq d. \quad (3.6)$$

Theorem 3.4. Let $(T(t))_{t \geq 0}$ be a bounded (C_0) -semigroup on a Banach space X and let A be its generator. Assume that $\sigma(A) \cap i\mathbb{R} \subseteq \bigcup_{j=1}^n \{ip_j\}$ for some $n \in \mathbb{N}$ and $\{p_j: 1 \leq j \leq n\} \subseteq \mathbb{R}$. Then, there exist positive constants C and T such that for all $t \geq T$,

$$\left\| T(t)(1 - A)^{-1} \prod_{j=1}^n (ip_j - A)(1 - A)^{-1} \right\| \leq C \left(m_{\log}^{-1}(t/4) + \frac{1}{M_{\log}^{-1}(t/4)} + \frac{1}{t} \right)$$

where M_{\log} and m_{\log} are defined as in (3.5) and (3.6), respectively.

In essence, the proof of this theorem is based on similar ideas to those in Theorem 2.3 and basic properties of C_0 -semigroups and infinitesimal generators, so we omit it.

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